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A certain class of dual equations involving series of Jacobi and Laguerre polynomials*

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SUMMARY

By employing an interesting modification of the familiar multiplying-factor technique, which was developed elsewhere by B. Noble [Proc. Cambridge Philos. Soc. **59**, 363–371 (1963)], an exact solution is obtained for a certain pair of dual equations involving series of Jacobi polynomials. Also computed are the values of these general series on the intervals over which their values are not already specified. Several special or confluent cases of the dual series equations considered here are shown to lead to (known or new) dual equations involving series of Jacobi or Laguerre polynomials; these simpler pairs of dual series equations were solved in the earlier works by A. P. Dwivedi and T. N. Trivedi [Indag. Math. **36**, 203–210 (1974)], J. S. Lowndes [Proc. Edinburgh Math. Soc. Ser. II **16**, 273–280 (1969)], Rekha Panda [Indag. Math. **39**, 122–127 (1977)], and H. M. Srivastava [*cf.*, *e.g.*, Indag. Math. **35**, 137–141 (1973)].

1. INTRODUCTION

Put

(1.1) $\quad I_1=\{x|0\leqq x<y\}$ and $I_2=\{x|y<x\leqq c\}$,

where c is an arbitrary positive number and $0<y<c$. Also, in Szegö's notation (*cf.* [10], p. 58), let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial of

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order (α, β) and degree n in x, defined by [*op. cit.*, p. 68]

$$(1.2) \qquad P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}, \quad n \geqq 0.$$

In the present paper we consider the problem of determining the sequence $\{A_n\}$ satisfying the dual series equations

$$(1.3) \qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\gamma+n+l+1)}{\Gamma(\varrho+n+l+1)} P_{n+l}^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right) = f(x), \quad \forall x \in I_1$$

and

$$(1.4) \qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\delta+n+l+1)}{\Gamma(\sigma+n+l+1)} P_{n+l}^{(\lambda,\mu)}\left(1-\frac{2x}{c}\right) = g(x), \quad \forall x \in I_2,$$

where $c > 0$, l is an arbitrary non-negative integer, $f(x)$ and $g(x)$ are prescribed functions, and in general,

$$(1.5) \qquad \min\{\alpha, \beta, \gamma, \delta, \lambda, \mu, \varrho, \sigma\} > -1.$$

The method employed is an interesting modification of the familiar multiplying-factor technique developed by Noble (*cf.* [4], p. 366, § 3) for solving a certain special pair of dual series equations in which the kernels involve Jacobi polynomials of the *same* order; indeed, Noble [*loc. cit.*] used Jacobi's original notation:

$$(1.6) \qquad \mathfrak{F}_n(a, \lambda; \varrho) = \binom{n+\lambda-1}{n}^{-1} P_n^{(\lambda-1, a-\lambda)}(1-2\varrho).$$

We also discuss the possibility of computing the values of the series (1.3) and (1.4) on the intervals I_2 and I_1, respectively, that is, on those intervals over which their values are not already specified; these values are of particular interest in various physical applications.

The dual series equations (1.3) and (1.4) are a generalization of those considered earlier by Lowndes [3], who (like Noble [4]) used Jacobi's notation (1.6), and by Dwivedi and Trivedi [1]. Furthermore, by suitably appealing to the principle of confluence exhibited in the known relationship [10, p. 103, Eq. (5.3.4)]

$$(1.7) \qquad \lim_{\beta\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) \right\} = L_n^{(\alpha)}(x),$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomial defined by

$$(1.8) \qquad L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \quad n = 0, 1, 2, \ldots,$$

our results in this paper would readily yield the corresponding results for dual equations involving series of Laguerre polynomials, which were given recently by Panda [5]. {See also Srivastava [7], [8] and [9].}

2. PRELIMINARY RESULTS

For the sake of ready reference we list here the following results involving Jacobi polynomials, which will be required in the course of our investigation.

(i) The orthogonality property of the Jacobi polynomials in the (slightly modified) form [10, p. 68, Eq. (4.3.3)]:

$$(2.1)\quad \begin{cases} \int_0^c x^\alpha \left(1-\frac{x}{c}\right)^\beta P_m^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right) P_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right) dx \\ = \frac{c^{\alpha+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}\,\delta_{mn},\ \alpha>-1,\ \beta>-1, \end{cases}$$

where δ_{mn} is the Kronecker delta.

(ii) The following forms of the known *fractional* integrals [2, p. 191, Eqs. (43) and (44)]:

$$(2.2)\quad \begin{cases} \int_0^\xi x^\alpha (\xi-x)^{\mu-1} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right) dx \\ = B(\alpha+n+1,\mu)\,\xi^{\alpha+\mu}\,P_n^{(\alpha+\mu,\,\beta-\mu)}\left(1-\frac{2\xi}{c}\right),\ \alpha>-1,\ \mu>0, \end{cases}$$

and

$$(2.3)\quad \begin{cases} \int_\xi^c \left(1-\frac{x}{c}\right)^\beta (x-\xi)^{\nu-1} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right) dx \\ = c^\nu B(\beta+n+1,\nu)\left(1-\frac{\xi}{c}\right)^{\beta+\nu} P_n^{(\alpha-\nu,\,\beta+\nu)}\left(1-\frac{2\xi}{c}\right),\ \beta>-1,\ \nu>0, \end{cases}$$

where $B(\alpha,\beta)$ denotes the familiar beta function defined by

$$(2.4)\quad B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ \alpha>0,\ \beta>0.$$

(iii) For integer $m\geqq 0$, we have the derivative formulas

$$(2.5)\quad \begin{cases} D_x^m\left\{x^{\alpha+m} P_n^{(\alpha+m,\,\beta-m)}\left(1-\frac{2x}{c}\right)\right\} \\ = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)}\,x^\alpha P_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right),\ D_x=d/dx, \end{cases}$$

and

$$(2.6)\quad \begin{cases} D_x^m\left\{\left(1-\frac{x}{c}\right)^{\beta+m} P_n^{(\alpha-m,\,\beta+m)}\left(1-\frac{2x}{c}\right)\right\} \\ = \frac{\Gamma(\beta+m+n+1)}{(-c)^m\,\Gamma(\beta+n+1)}\left(1-\frac{x}{c}\right)^\beta P_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right). \end{cases}$$

PROOF OF (2.5). Making use of the known results (7), p. 264 and (17), p. 265 in reference [6], we observe that

$$(2.7)\qquad (x-1)D_x\left\{P_n^{(\alpha,\beta)}(x)\right\}=(\alpha+n)\,P_n^{(\alpha-1,\beta+1)}(x)-\alpha P_n^{(\alpha,\beta)}(x),$$

whence

$$(2.8)\qquad D_x\left\{x^\alpha\,P_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right)\right\}=(\alpha+n)x^{\alpha-1}\,P_n^{(\alpha-1,\beta+1)}\left(1-\frac{2x}{c}\right),$$

and the derivative formula (2.5) follows by induction.

PROOF OF (2.6). By combining the known results (5), p. 264 and (17), p. 265 in reference [6], we get

$$(2.9)\qquad (x+1)D_x\left\{P_n^{(\alpha,\beta)}(x)\right\}=(\beta+n)\,P_n^{(\alpha+1,\beta-1)}(x)-\beta\,P_n^{(\alpha,\beta)}(x),$$

which readily yields

$$(2.10)\qquad \begin{cases} D_x\left\{\left(1-\dfrac{x}{c}\right)^\beta P_n^{(\alpha,\beta)}\left(1-\dfrac{2x}{c}\right)\right\}\\ =-c^{-1}(\beta+n)\left(1-\dfrac{x}{c}\right)^{\beta-1}P_n^{(\alpha+1,\beta-1)}\left(1-\dfrac{2x}{c}\right),\end{cases}$$

and the derivative formula (2.6) would follow, as before, by induction on the non-negative integer m.

REMARK 1. In view of the known identity [10, p. 59, Eq. (4.1.3)]

$$(2.11)\qquad P_n^{(\alpha,\beta)}(-x)=(-1)^n P_n^{(\beta,\alpha)}(x),$$

it is easily verified that the integral formulas (2.2) and (2.3) are essentially equivalent, and so also are the derivative formulas (2.5) and (2.6).

3. THE MODIFIED MULTIPLYING-FACTOR TECHNIQUE

In order to illustrate the modified multiplying-factor technique, we now turn to our series equation (1.3), multiply it by $x^\alpha(\xi-x)^{m+p-1}$, where p is an arbitrary constant and m is a suitable non-negative integer, and integrate both sides with respect to x over the interval $(0,\xi)$. By applying the integral formula (2.2), we thus obtain

$$(3.1)\qquad \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n\,\frac{\Gamma(\gamma+n+l+1)\Gamma(\alpha+n+l+1)}{\Gamma(\varrho+n+l+1)\Gamma(\alpha+n+l+m+p+1)}\,P_{n+l}^{(\alpha+m+p,\,\beta-m-p)}\left(1-\frac{2\xi}{c}\right)\\ \displaystyle =\frac{\xi^{-\alpha-m-p}}{\Gamma(m+p)}\int_0^\xi x^\alpha\,(\xi-x)^{m+p-1}f(x)\,dx,\end{cases}$$

where $0<\xi<y$, $\alpha>-1$, and $m+p>0$.

If we multiply (3.1) by $\xi^{\alpha+m+p}$ and differentiate the resulting equation m times with respect to ξ, using the derivative formula (2.5), we shall get

$$\text{(3.2)}\qquad \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n \frac{\Gamma(\gamma+n+l+1)\Gamma(\alpha+n+l+1)}{\Gamma(\varrho+n+l+1)\Gamma(\alpha+n+l+p+1)} P_{n+l}^{(\alpha+p,\,\beta-p)}\left(1-\frac{2\xi}{c}\right) \\ \displaystyle = \frac{\xi^{-\alpha-p}}{\Gamma(m+p)} D_\xi^m \left\{ \int_0^\xi x^\alpha\,(\xi-x)^{m+p-1} f(x)\,dx \right\}, \end{cases}$$

where, as before, $0<\xi<y$, $\alpha>-1$, and $m+p>0$.

Next we multiply our series equation (1.4) by $(1-x/c)^\mu$ and differentiate the resulting equation k times with respect to x, using the derivative formula (2.6). We thus have

$$\text{(3.3)}\qquad \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n \frac{\Gamma(\delta+n+l+1)\Gamma(\mu+n+l+1)}{\Gamma(\sigma+n+l+1)\Gamma(\mu+n+l-k+1)} P_{n+l}^{(\lambda+k,\,\mu-k)}\left(1-\frac{2x}{c}\right) \\ \displaystyle =(-c)^k\left(1-\frac{x}{c}\right)^{k-\mu} D_x^k\left\{\left(1-\frac{x}{c}\right)^\mu g(x)\right\},\ \forall x\in I_2, \end{cases}$$

where k is a non-negative integer.

Multiplying this last equation (3.3) by $(1-x/c)^{\mu-k}\,(x-\xi)^{q+k-1}$, where q is an arbitrary constant, if we integrate both sides with respect to x over the interval (ξ, c) using (2.3), we shall obtain

$$\text{(3.4)}\qquad \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n \frac{\Gamma(\delta+n+l+1)\Gamma(\mu+n+l+1)}{\Gamma(\sigma+n+l+1)\Gamma(\mu+n+l+q+1)} P_{n+l}^{(\lambda-q,\,\mu+q)}\left(1-\frac{2\xi}{c}\right) \\ \displaystyle = \frac{(-1)^k c^{-q}}{\Gamma(q+k)}\left(1-\frac{\xi}{c}\right)^{-\mu-q} \int_\xi^c (x-\xi)^{q+k-1} D_x^k\left\{\left(1-\frac{x}{c}\right)^\mu g(x)\right\}dx, \end{cases}$$

where $y<\xi<c$, $\mu-k>-1$, and $q+k>0$.

The modified multiplying-factor technique applies successfully to those situations in which the arbitrary constants p and q can be so chosen that the left-hand sides of the transformed equations (3.2) and (3.4) become identical; indeed, in these (successful) cases, the unknown coefficients A_n can be determined from (3.2) and (3.4) by merely appealing to the orthogonality relationship (2.1). In the next two sections we shall present a systematic discussion of all such situations of interest.

4. THE SPECIAL CASE $\varrho=\alpha$ AND $\sigma=\mu$

In the special case when $\varrho=\alpha$ and $\sigma=\mu$, if we set $p=\gamma-\alpha$ and $q=\delta-\mu$, the left-hand sides of equations (3.2) and (3.4) do indeed become identical provided that

$$\text{(4.1)}\qquad \alpha+\beta=\gamma+\delta=\lambda+\mu,$$

and by using the orthogonality property (2.1) we readily obtain our first set of results contained in

THEOREM 1. *For $c>0$, let the sequence $\{A_n\}$ be defined by the dual series equations*

$$(4.2)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\gamma+n+l+1)}{\Gamma(\alpha+n+l+1)} P_{n+l}^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right)=f(x),\ \ \forall x\in I_1$$

and

$$(4.3)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\delta+n+l+1)}{\Gamma(\mu+n+l+1)} P_{n+l}^{(\lambda,\mu)}\left(1-\frac{2x}{c}\right)=g(x),\ \ \forall x\in I_2,$$

where l is an arbitrary non-negative integer, and the intervals I_1 and I_2 are given by (1.1).

Then, for integers $k, m, n \geqq 0$,

$$(4.4)\qquad \begin{cases} A_n=\dfrac{(n+l)!\,(\alpha+\beta+2n+2l+1)\Gamma(\alpha+\beta+n+l+1)}{c^{\gamma+1}\,\Gamma(\gamma+n+l+1)\Gamma(\delta+n+l+1)} \\ \cdot\left[\dfrac{1}{\Gamma(\gamma-\alpha+m)}\displaystyle\int_0^{y}\left(1-\frac{\xi}{c}\right)^{\delta} P_{n+l}^{(\gamma,\delta)}\left(1-\frac{2\xi}{c}\right)F(\xi)\,d\xi\right. \\ \left.+\dfrac{(-1)^k c^{\gamma-\lambda}}{\Gamma(\lambda-\gamma+k)}\displaystyle\int_y^{c}\xi^{\gamma} P_{n+l}^{(\gamma,\delta)}\left(1-\frac{2\xi}{c}\right)G(\xi)\,d\xi\right], \end{cases}$$

where, for convenience,

$$(4.5)\qquad F(\xi)=D_{\xi}^{m}\left\{\int_0^{\xi} x^{\alpha}(\xi-x)^{\gamma-\alpha+m-1}f(x)\,dx\right\}$$

and

$$(4.6)\qquad G(\xi)=\int_{\xi}^{c}(x-\xi)^{\lambda-\gamma+k-1}D_x^k\left\{\left(1-\frac{x}{c}\right)^{\mu}g(x)\right\}dx,$$

provided that (4.1) *holds and*

$$(4.7)\qquad \alpha+\beta+1>\gamma>-1,\ \ \gamma+m>\alpha>-1,\ \ \delta>\mu-k>-1.$$

REMARK 2. For $l=0$, and under the parametric constraints in (4.1), the dual series equations (4.2) and (4.3) would reduce essentially to those considered earlier by Dwivedi and Trivedi [1], and indeed our solution (4.4) with $l=0$ can be shown fairly easily to be in complete agreement with the Dwivedi-Trivedi solution [*op. cit.*, p. 207, Eq. (3.13)]. {See also Noble [4] for a further special case of Theorem 1.}

A confluent case of Theorem 1 is worthy of note. Indeed, it would follow if we replace β and μ by their values in terms of δ, given by (4.1), set $\delta=c$, $A_n=B_n/\Gamma(\gamma+n+l+1)$, replace $g(x)$ by $c^{\lambda-\gamma}g(x)$, and take the limits of equations (4.2), (4.3), (4.4) and (4.6) when $c\to\infty$. Making use of the known relationship (1.7), we thus have

THEOREM 2. *The solution of the dual Laguerre series equations*

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma(\alpha+n+l+1)} L_{n+l}^{(\alpha)}(x) = f(x),\ 0 \leq x < y \tag{4.8}$$

and

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma(\gamma+n+l+1)} L_{n+l}^{(\lambda)}(x) = g(x),\ y < x < \infty, \tag{4.9}$$

is given by

$$\begin{cases} B_n = \dfrac{(n+l)!}{\Gamma(\gamma-\alpha+m)} \displaystyle\int_0^y e^{-\xi} L_{n+l}^{(\gamma)}(\xi) F(\xi)\, d\xi \\ + \dfrac{(-1)^k (n+l)!}{\Gamma(\lambda-\gamma+k)} \displaystyle\int_y^\infty \xi^\gamma L_{n+l}^{(\gamma)}(\xi) K(\xi)\, d\xi, \end{cases} \tag{4.10}$$

where $k, l, m, n \in \{0, 1, 2, \ldots\}$, $F(\xi)$ *is given by* (4.5), *and*

$$K(\xi) = \int_\xi^\infty (x-\xi)^{\lambda-\gamma+k-1} D_x^k \left\{ e^{-x} g(x) \right\} dx, \tag{4.11}$$

provided that $\gamma+m > \alpha > -1$ *and* $\lambda+k > \gamma > -1$.

REMARK 3. For $k=0$, if we replace λ by σ, and l by p, and set $\gamma=\alpha+\beta-1$, Theorem 2 would evidently yield the main result of a recent paper by Panda [5, p. 124, Eq. (11)]. {For other known special cases of Theorem 2, the interested reader may be referred to [5, p. 127, Remarks 1 and 2] and Srivastava's papers [7], [8] and [9].}

5. THE SPECIAL CASE $\varrho=\beta$ AND $\sigma=\lambda$

When $\varrho=\beta$ and $\sigma=\lambda$, if we choose $p=\gamma-\alpha$ and $q=\delta-\mu$, the transformed equations (3.2) and (3.4) would reduce to

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n \frac{\Gamma(\alpha+n+l+1)}{\Gamma(\beta+n+l+1)} P_{n+l}^{(\gamma,\alpha+\beta-\gamma)}\left(1-\frac{2\xi}{c}\right) \\ = \dfrac{\xi^\gamma}{\Gamma(\gamma-\alpha+m)} D_\xi^m \left\{ \displaystyle\int_0^\xi x^\alpha (\xi-x)^{\gamma-\alpha+m-1} f(x)\, dx \right\}, \end{cases} \tag{5.1}$$

where $0<\xi<y$ and $\gamma+m>\alpha>-1$, and

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu+n+l+1)}{\Gamma(\alpha+n+l+1)} P_{n+l}^{(\lambda+\mu-\delta,\delta)}\left(1-\frac{2\xi}{c}\right) \\ = \dfrac{(-1)^k c^{\mu-\delta}}{\Gamma(\delta-\mu+k)} \left(1-\dfrac{\xi}{c}\right)^{-\delta} \displaystyle\int_\xi^c (x-\xi)^{\delta-\mu+k-1} D_x^k \left\{ \left(1-\frac{x}{c}\right)^\mu g(x) \right\} dx, \end{cases} \tag{5.2}$$

where $y<\xi<c$ and $\delta>\mu-k>-1$, k and m being non-negative integers.

The left-hand sides of these reduced equations (5.1) and (5.2) cannot be made identical except possibly in the seemingly trivial case when, in addition to the parametric constraints in (4.1), we have $\mu=\alpha=\beta$. Nonetheless, if we let $\gamma=\beta+\nu$ and $\delta=\alpha-\nu$, our dual series equations (1.3) and (1.4) would reduce in this case to

$$(5.3)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\beta+\nu+n+l+1)}{\Gamma(\beta+n+l+1)} P_{n+l}^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right)=f(x),\ \forall x\in I_1$$

and

$$(5.4)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\alpha-\nu+n+l+1)}{\Gamma(\alpha+n+l+1)} P_{n+l}^{(\lambda,\mu)}\left(1-\frac{2x}{c}\right)=g(x),\ \forall x\in I_2,$$

respectively.

For $l=0$, $\lambda=\alpha$ and $\mu=\beta$, these last dual series equations (5.3) and (5.4) are essentially the same as those solved earlier by Lowndes [3] under the conditions:

$$(5.5)\qquad \begin{cases} \text{(i) } \alpha+\beta+1>\alpha>\nu-1,\ 0<\nu<1;\ \text{or} \\ \text{(ii) } \alpha+\beta+\nu+1>\alpha>-1,\ -1<\nu<0. \end{cases}$$

When $l\geqq 0$, by appealing to the transformed equations (5.1) and (5.2), we may state

THEOREM 3. *For integers k, $m\geqq 0$, let*

$$(5.6)\qquad f^*(x)=\frac{x^{\beta+\nu}}{\Gamma(\beta-\alpha+\nu+m)} D_x^m\left\{\int_0^x t^\alpha (x-t)^{\beta-\alpha+\nu+m-1} f(t)dt\right\}$$

and

$$(5.7)\qquad \begin{cases} g^*(x)=\dfrac{(-1)^k c^{\beta-\alpha+\nu}}{\Gamma(\alpha-\beta-\nu+k)}\left(1-\dfrac{x}{c}\right)^{\nu-\alpha} \\ \cdot \displaystyle\int_x^c (t-x)^{\alpha-\beta-\nu+k-1} D_t^k\left\{\left(1-\frac{t}{c}\right)^\beta g(t)\right\}dt, \end{cases}$$

where $\beta+\nu+m>\alpha>-1$ and $\alpha-\nu>\beta-k>-1$.

Then the solution of the pair of dual Jacobi series equations (5.3) *and* (5.4) *with $\lambda=\alpha$ and $\mu=\beta$ is the same as that of the pair:*

$$(5.8)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\alpha+n+l+1)}{\Gamma(\beta+n+l+1)} P_{n+l}^{(\beta+\nu,\alpha-\nu)}\left(1-\frac{2x}{c}\right)=f^*(x),\ \forall x\in I_1$$

and

$$(5.9)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(\beta+n+l+1)}{\Gamma(\alpha+n+l+1)} P_{n+l}^{(\beta+\nu,\alpha-\nu)}\left(1-\frac{2x}{c}\right)=g^*(x),\ \forall x\in I_2,$$

where l is any non-negative integer, and the intervals I_1 and I_2 are defined by (1.1).

6. UNSPECIFIED VALUES OF SERIES (1.3) AND (1.4)

The values of the series (1.3) and (1.4) are not specified in the intervals I_2 and I_1, respectively. Since these quantities are of particular interest in many physical applications, we shall now show how each of them can be determined in case of the dual Jacobi series equations (4.2) and (4.3).

We begin by assuming that

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\gamma+n+l+1)}{\Gamma(\alpha+n+l+1)} P_{n+l}^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right)=\phi(x), \quad \forall x \in I_2 \tag{6.1}$$

and

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\delta+n+l+1)}{\Gamma(\mu+n+l+1)} P_{n+l}^{(\lambda,\mu)}\left(1-\frac{2x}{c}\right)=\psi(x), \quad \forall x \in I_1, \tag{6.2}$$

where I_1 and I_2 are given by (1.1), and the various parameters are constrained as in (4.1).

Making use of the derivative formula (2.5), equation (6.1) can be rewritten in the form

$$\phi(x)=x^{-\alpha} D_x^r\left\{x^{\alpha+r} \sum_{n=0}^{\infty} A_n \frac{\Gamma(\gamma+n+l+1)}{\Gamma(\alpha+n+l+r+1)} P_{n+l}^{(\alpha+r,\beta-r)}\left(1-\frac{2x}{c}\right)\right\}, \tag{6.3}$$

where $x \in I_2$ and r is a non-negative integer.

If we substitute for the coefficients A_n from (4.4) into this last equation (6.3), we get

$$\begin{cases} \phi(x)=x^{-\alpha} D_x^r\left\{\dfrac{x^{\alpha+r}}{\Gamma(\gamma-\alpha+m)} \displaystyle\int_0^v \left(1-\frac{\xi}{c}\right)^{\delta} P(x,\xi)\, F(\xi)\, d\xi\right. \\ \left. + \dfrac{(-1)^k c^{\gamma-\lambda} x^{\alpha+r}}{\Gamma(\lambda-\gamma+k)} \displaystyle\int_v^c \xi^{\gamma} P(x,\xi)\, G(\xi)\, d\xi\right\}, \quad \forall x \in I_2, \end{cases} \tag{6.4}$$

where, for convenience,

$$P(x,\xi)=\sum_{n=0}^{\infty} \Delta_n(x,\xi)-M(x,\xi), \tag{6.5}$$

$$\begin{cases} \Delta_n(x,\xi)=\dfrac{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{c^{\gamma+1}\Gamma(\alpha+n+r+1)\Gamma(\delta+n+1)} \\ \cdot P_n^{(\alpha+r,\beta-r)}\left(1-\dfrac{2x}{c}\right) P_n^{(\gamma,\delta)}\left(1-\dfrac{2\xi}{c}\right) \end{cases} \tag{6.6}$$

and

$$M(x,\xi)=\sum_{n=0}^{l-1} \Delta_n(x,\xi), \tag{6.7}$$

it being understood that $M(x,\xi)=0$ when $l=0$.

The infinite series on the right-hand side of (6.5) can be summed (at least formally) by appealing to the orthogonality relationship (2.1) and the integral formula (2.2), and we thus have

$$(6.8)\qquad \sum_{n=0}^{\infty} \Delta_n(x, \xi) = \frac{(x-\xi)^{\alpha-\gamma+r-1}}{\Gamma(\alpha-\gamma+r)} x^{-\alpha-r} \left(1-\frac{\xi}{c}\right)^{-\delta} H(x-\xi),$$

where $H(t)$ denotes Heaviside's unit function, and

$$(6.9)\qquad \alpha-\gamma+r>0,\ \beta-r>-1.$$

Substituting from (6.8) into (6.5), and then into (6.4), we finally get one of the desired values, *viz*

$$(6.10)\qquad \begin{cases} \phi(x) = \dfrac{x^{-\alpha}}{\Gamma(\alpha-\gamma+r)} D_x^r \left\{ \dfrac{1}{\Gamma(\gamma-\alpha+m)} \displaystyle\int_0^y (x-\xi)^{\alpha-\gamma+r-1} F(\xi)\, d\xi \right. \\ \left. + \dfrac{(-1)^k c^{\gamma-\lambda}}{\Gamma(\lambda-\gamma+k)} \displaystyle\int_y^x \xi^\gamma (x-\xi)^{\alpha-\gamma+r-1} \left(1-\frac{\xi}{c}\right)^{-\delta} G(\xi)\, d\xi \right\} \\ -x^{-\alpha} D_x^r \left\{ \dfrac{x^{\alpha+r}}{\Gamma(\gamma-\alpha+m)} \displaystyle\int_0^y \left(1-\frac{\xi}{c}\right)^{\delta} M(x, \xi) F(\xi)\, d\xi \right. \\ \left. + \dfrac{(-1)^k c^{\gamma-\lambda} x^{\alpha+r}}{\Gamma(\lambda-\gamma+k)} \displaystyle\int_y^c \xi^\gamma M(x, \xi) G(\xi)\, d\xi \right\}, \quad \forall x \in I_2, \end{cases}$$

where $F(\xi)$, $G(\xi)$ and $M(x, \xi)$ are given by (4.5), (4.6) and (6.7), respectively, and the parametric constraints in (4.1), (4.7) and (6.9) are assumed to hold, k, m and r being non-negative integers.

In order to determine the value of $\psi(x)$, defined by (6.2), we first apply the derivative formula (2.6) and rewrite (6.2) as

$$(6.11)\qquad \begin{cases} \psi(x) = \left(1-\dfrac{x}{c}\right)^{-\mu} (-c\, D_x)^s \left\{ \left(1-\dfrac{x}{c}\right)^{\mu+s} \right. \\ \left. \cdot \displaystyle\sum_{n=0}^{\infty} A_n \frac{\Gamma(\delta+n+l+1)}{\Gamma(\mu+n+l+s+1)} P_{n+l}^{(\lambda-s,\mu+s)}\left(1-\frac{2x}{c}\right) \right\}, \quad \forall x \in I_1, \end{cases}$$

where s is a non-negative integer.

Now we substitute for the coefficients A_n from (4.4) into (6.11), and change the order of integration and summation. We thus find that

$$(6.12)\qquad \begin{cases} \psi(x) = \left(1-\dfrac{x}{c}\right)^{-\mu} (-c\, D_x)^s \left\{ \dfrac{(1-x/c)^{\mu+s}}{\Gamma(\gamma-\alpha+m)} \displaystyle\int_0^y \left(1-\frac{\xi}{c}\right)^{\delta} Q(x, \xi) F(\xi)\, d\xi \right. \\ \left. + \dfrac{(-1)^k c^{\gamma-\lambda} (1-x/c)^{\mu+s}}{\Gamma(\lambda-\gamma+k)} \displaystyle\int_y^c \xi^\gamma Q(x, \xi) G(\xi)\, d\xi \right\}, \quad \forall x \in I_1, \end{cases}$$

where, for convenience,

$$(6.13)\qquad Q(x, \xi) = \sum_{n=0}^{\infty} \Omega_n(x, \xi) - N(x, \xi),$$

$$(6.14)\quad \begin{cases} \Omega_n(x,\xi)=\dfrac{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{c^{\gamma+1}\Gamma(\gamma+n+1)\Gamma(\mu+n+s+1)} \\ \cdot P_n^{(\lambda-s,\mu+s)}\left(1-\dfrac{2x}{c}\right)P_n^{(\gamma,\delta)}\left(1-\dfrac{2\xi}{c}\right) \end{cases}$$

and

$$(6.15)\quad N(x,\xi)=\sum_{n=0}^{l-1}\Omega_n(x,\xi),$$

it being understood, as before, that this last sum is nil unless $l\geqq 1$.

By applying the orthogonality property (2.1) and the integral formula (2.3), it is easily seen that

$$(6.16)\quad \sum_{n=0}^{\infty}\Omega_n(x,\xi)=\frac{(\xi-x)^{\gamma-\lambda+s-1}}{\Gamma(\gamma-\lambda+s)}c^{\lambda-\gamma-s}\xi^{-\gamma}\left(1-\frac{x}{c}\right)^{-\mu-s}H(\xi-x),$$

where $\Omega_n(x,\xi)$ is defined by (6.14), and

$$(6.17)\quad \gamma>\lambda-s>-1.$$

Substituting from (6.16) into (6.13), and then into (6.12), we finally obtain the other desired value given by

$$(6.18)\quad \begin{cases} \psi(x)=\dfrac{c^{\lambda-\gamma}(1-x/c)^{-\mu}}{\Gamma(\gamma-\lambda+s)}(-D_x)^s\left\{\dfrac{1}{\Gamma(\gamma-\alpha+m)}\displaystyle\int_x^v\xi^{-\gamma}\left(1-\frac{\xi}{c}\right)^{\delta}(\xi-x)^{\gamma-\lambda+s-1}\right. \\ \left.\cdot F(\xi)\,d\xi+\dfrac{(-1)^k c^{\gamma-\lambda}}{\Gamma(\lambda-\gamma+k)}\displaystyle\int_v^c(\xi-x)^{\gamma-\lambda+s-1}G(\xi)\,d\xi\right\} \\ -\left(1-\dfrac{x}{c}\right)^{-\mu}(-c\,D_x)^s\left\{\dfrac{(1-x/c)^{\mu+s}}{\Gamma(\gamma-\alpha+m)}\displaystyle\int_0^v\left(1-\frac{\xi}{c}\right)^{\delta}N(x,\xi)F(\xi)\,d\xi\right. \\ \left.+\dfrac{(-1)^k c^{\gamma-\lambda}(1-x/c)^{\mu+s}}{\Gamma(\lambda-\gamma+k)}\displaystyle\int_v^c\xi^{\gamma}N(x,\xi)G(\xi)\,d\xi\right\},\quad \forall x\in I_1, \end{cases}$$

where $F(\xi)$, $G(\xi)$ and $N(x,\xi)$ are defined by (4.5), (4.6) and (6.15), respectively, and the parameters are constrained as in (4.1), (4.7) and (6.17), k, m and s being non-negative integers.

Evidently, when $l=0$, our results (6.10) and (6.18) would simplify considerably, and we shall be led eventually to those given earlier by Dwivedi and Trivedi [1, p. 209, Eqs. (4.5) and (4.8)]. Moreover, by suitably appealing to the principle of confluence (which indeed was used above in deriving Theorem 2 from Theorem 1) it is easy to observe how the confluent forms of equations (6.10) and (6.18) will yield the values of the Laguerre series (4.8) and (4.9) on the intervals over which their values are not already prescribed. In fact, for $r=0$, the values thus obtained are in complete agreement with the corresponding results given recently by Panda [1] [5, p. 125, Eq. (20); p. 127, Eq. (27)]. We omit the details involved.

Theorems 1 and 3 (and, of course, the results of this section) can be appropriately specialized to hold for the dual equations involving series of the Gegenbauer (or ultraspherical) polynomials

$$C_n^{\alpha+\frac{1}{2}}(x)=\binom{n+\alpha}{n}^{-1}\binom{n+2\alpha}{n}P_n^{(\alpha,\alpha)}(x), \tag{6.19}$$

the relatively more familiar Legendre polynomials

$$P_n(x)=P_n^{(0,0)}(x), \tag{6.20}$$

the Tchebycheff polynomials (of the first and second kinds)

$$\begin{cases} T_n(x)=\binom{n-\frac{1}{2}}{n}^{-1}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x), \\ U_n(x)=\frac{1}{2}\binom{n+\frac{1}{2}}{n+1}^{-1}P_n^{(\frac{1}{2},\frac{1}{2})}(x), \end{cases} \tag{6.21}$$

and of several other classes of orthogonal polynomials which are special cases of the Jacobi polynomials. On the other hand, results involving dual Hermite series equations can be deduced as a confluent case of Theorem 2, since it is fairly well known that [10, p. 389, Problem 80]

$$H_n(x/\sqrt{2})=(-1)^n\,2^{n/2}\,n!\lim_{\alpha\to\infty}\left\{\alpha^{-n/2}L_n^{(\alpha)}(\alpha+x\sqrt{\alpha})\right\}. \tag{6.22}$$

Alternatively, the solution of dual equations involving series of Hermite polynomials can be deduced from the results of this paper by appealing to the familiar relationship [*op. cit.*, p. 106, Eq. (5.6.1)]

$$H_{2n+\varepsilon}(x)=(-1)^n\,2^{2n+\varepsilon}\,n!\,x^{\varepsilon}L_n^{(\varepsilon-\frac{1}{2})}(x^2),\quad \varepsilon=0 \text{ or } \varepsilon=1, \tag{6.23}$$

or to the principle of confluence exhibited by

$$H_n(x)=n!\lim_{\alpha\to\infty}\left\{\alpha^{-n/2}C_n^{\alpha}(x/\sqrt{\alpha})\right\}, \tag{6.24}$$

where $C_n^{\alpha}(x)$ are the Gegenbauer polynomials defined by (6.19).

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[1] Notice here that the upper limit of the second integral in equation (20) on page 125, as well as the lower limit of the first integral in equation (27) on page 127, of Panda's paper [5] should be corrected to read x.

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